Non-Abelian Anomalies and Effective Actions for a Homogeneous Space G/H

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Abstract

We consider the problem of constructing the fully gauged effective action in $2n$-dimensional space-time for Nambu-Goldstone bosons valued in a homogeneous space G/H, with the requirement that the action be a solution of the anomalous Ward identity and be invariant under the gauge transformations of H. We show that this can be done whenever the homotopy group $\pi_{2n}(G/H)$ is trivial, G/H is reductive and H is embedded in G so as to be anomaly free, in particular if H is an anomaly safe group. We construct the necessary generalization of the Bardeen counterterm and give explicit forms for the anomaly and the effective action. When G/H is a symmetric space the counterterm and the anomaly decompose into a parity even and a parity odd part. In this case, for the parity even part of the action, one does not need the anomaly free embedding of H.

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1 Introduction

Recently there has been renewed interest [1, 2] in the study of effective actions which describe the interactions of Nambu-Goldstone bosons due to the existence of non-Abelian anomalies. The general case in which the Nambu-Goldstone fields are valued in a coset space G/H corresponding to a reductive homogeneous space has been analyzed in the past. Here G is a Lie group and H a subgroup of it. In reference [3] the action was not fully gauged, in reference [4] it was but the formulas for the anomaly and the action were not made totally explicit. In the present paper we reconsider the question of constructing the general fully gauged action with the aim of providing simple derivations of the general formulas and of making all expressions as explicit as possible in the case of four space-time dimensions.

In section 2.1 we recall the consistency condition satisfied by the non-Abelian anomaly and the construction of the effective action in $2n$-dimensional space-time from the anomaly [5]. This construction is valid for the general reductive case when the anomaly vanishes for the currents of the subgroup H, but for arbitrary external gauge fields in G. In section 2.2 we give the well known expression of the Chern-Simon form and of the "canonical anomaly".

In general, the canonical anomaly for the group G does not have the property that the currents of the subgroup H are anomaly free, and therefore cannot be used as it stands to apply the construction of the effective action described in section 2.1. If the group H is embedded in G in an anomaly free way, in particular if H is anomaly safe [‡], one can find a counterterm to add to the action which modifies the anomaly so that it has the above mentioned property. We refer to this as the "shifted" anomaly. This counterterm is a generalization of the counterterm used by Bardeen [7]. The general form of the counterterm and the expression for the shifted anomaly are given in

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[‡]Anomaly safe groups are groups for which no representation has anomalies [6]. In four dimensions, the only simple compact groups which have anomalies are the $SU(N)$ groups for $N \geq 3$ which includes $Spin(6) = SU(4)$. E_6 is also an anomaly safe group.

section 3. An explicit form for the counterterm is given for four space-time dimensions.

The special case when G/H is a symmetric space is discussed in section 4. Here there exists a "parity" isomorphism of the Lie algebra of G which permits the splitting of the counterterm and of the shifted anomaly into a parity even and a parity odd part. It is remarkable that the parity even part satisfies the requirement of conserved H currents even if H is not embedded in an anomaly free way. Both the parity even and the parity odd part can arise from fermion loops, but only the parity even part seems to be of physical interest. At the end of section 4, the complete shifted anomaly is given for the general (nonsymmetric) reductive case in four space-time dimensions.

In Appendix 1, we recall the definition and properties of the "homotopy operator" which was used in sections 2.2 and 3 to obtain expressions valid in any number of dimensions. The reader who is interested only in four dimensional space-time can easily check directly all formulas using the explicit expressions (27), (59) and following the arguments of section 4.

Finally, in Appendix 2, we give a more geometric description of the effective action, which follows the ideas developed in references [3, 4, 8, 9, 10]. It is well known [10] that the differential forms which give the anomalies and the effective actions must be suitably normalized so that the theory can be consistently quantized. In the present paper, we shall omit all normalization factors and overall factors of $\sqrt{-1}$ in order to write simpler formulas. These factors can always be reinstated and have been discussed for most cases of interest in [1, 8, 9, 10, 11, 12].

2 Non-Abelian Anomaly

In this section we review some basic facts about the non-Abelian anomaly.

Given a Lie group G and a Lie subgroup H, we denote their Lie algebras

by $\mathbf{g} = Lie(G)$, $\mathbf{h} = Lie(H)$. We split $\mathbf{g}$ as $\mathbf{g} = \mathbf{h} + \mathbf{k}$; if $\mathbf{h}$ and $\mathbf{k}$ satisfy

$$[\mathbf{h}, \mathbf{k}] \subseteq \mathbf{k}, \tag{1}$$

then G/H is said to be *reductive*. If further

$$[\mathbf{k}, \mathbf{k}] \subseteq \mathbf{h}, \tag{2}$$

then G/H is *symmetric*.

Notice that G/H admits a reductive splitting whenever H is compact. The reason is the following. Given the compact $\mathbf{h}$, we can find its orthogonal complement $\mathbf{k}$ such that $\mathbf{g} = \mathbf{h} + \mathbf{k}$ and $Tr(\lambda_i \lambda_a) = 0$ in any representation of $\mathbf{g}$ for $\lambda_i \in \mathbf{h}$ and $\lambda_a \in \mathbf{k}$. In general $[\lambda_i, \lambda_a] = f_{iaj}\lambda_j + f_{iab}\lambda_b$ for $\lambda_i, \lambda_j \in \mathbf{h}$ and $\lambda_a, \lambda_b \in \mathbf{k}$. The fact that f_{iaj} vanishes follows from the cyclic property of trace: $f_{iaj} = Tr([\lambda_i, \lambda_a]\lambda^j) = Tr([\lambda^j, \lambda_i]\lambda_a) = 0$, where λ^j is the dual of λ_j, namely, $Tr(\lambda_i \lambda^j) = \delta_i^j$. The dual basis of $\mathbf{h}$ exists because H is compact, which implies that the Killing metric of $\mathbf{h}$ is not degenerate.

2.1 Consistency Condition and Effective Action

In this paper we use Einstein's summation convention and the convention that for any $\alpha = \alpha^a(x)\lambda_a$ and $F_a(x)$

$$F_\alpha = \alpha \cdot F = \int (d^D x)\alpha^a(x)F_a(x), \tag{3}$$

where D could be $2n$ or $2n+1$ for $n = 1, 2, \cdots$.

In general, the effective action $W[\xi, A]$ of the Nambu-Goldstone boson fields $\xi(x) \in \mathbf{k}$ in the presence of external gauge fields $A_\mu = A_\mu^a(x)\lambda_a$, where the λ_a's are an anti-Hermitian matrix representation of the generators of $\mathbf{g}$, satisfies the anomalous Ward identity

$$\delta_\alpha W[\xi, A] = \int (d^{2n} x)\alpha^a(x)G_a[A](x) = G_\alpha[A], \tag{4}$$

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where $\delta_\alpha = Y_\alpha + Z_\alpha$, with Y_α acting only on A and Z_α only on ξ. The operators Y_α which generate the transformation of the gauge fields are

$$Y_a(x) = -\partial_\mu \frac{\delta}{\delta A_\mu^a(x)} - f_{abc} A_\mu^b(x) \frac{\delta}{\delta A_\mu^c(x)}. \tag{5}$$

It is convenient to use the language of differential forms. Introducing the matrix valued 1-form $A(x) = dx^\mu A_\mu$, we have

$$\delta_\alpha A = d\alpha + [A, \alpha], \tag{6}$$

$$\delta_\alpha F = [F, \alpha], \tag{7}$$

where $F = dA + A^2$ and the exterior differential operator $d = dx^\mu \partial_\mu$ commutes with δ_α.

The action of Z_α on the Nambu-Goldstone bosons which transform linearly under H and nonlinearly under G [13] is specified by the transformation

$$e^{-\xi'} = e^{Z_\alpha} e^{-\xi} = e^{-\alpha} e^{-\xi} e^\eta, \tag{8}$$

where $\xi \in \mathbf{k}$ and $\eta \in \mathbf{h}$ is chosen such that $\xi' \in \mathbf{k}$.

An important observation is that the anomaly $G_\alpha[A]$ must be "consistent", i.e. that it must satisfy the consistency condition

$$Y_{\alpha_1} G_{\alpha_2}[A] - Y_{\alpha_2} G_{\alpha_1}[A] = [\alpha_1, \alpha_2] \cdot G[A], \tag{9}$$

which is a direct result of (4) and the fact that δ_α generates the non-Abelian gauge transformations and satisfies

$$\delta_{\alpha_1} \delta_{\alpha_2} - \delta_{\alpha_2} \delta_{\alpha_1} - \delta_{[\alpha_1, \alpha_2]} = 0. \tag{10}$$

For a more complete explanation see [5].

Now we want to investigate the conditions under which one can integrate the anomalous Ward identity to get an effective action $W[\xi, A]$ satisfying (4). Given any consistent anomaly $G_\alpha[A]$, consider the functional $W[\xi, A]$ as in [5]

$$W[\xi, A] = \int_0^1 dt \xi \cdot G[A_t], \tag{11}$$

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where

$$A_t = e^{-tY_\xi} A = e^{t\xi} A e^{-t\xi} + e^{t\xi} d e^{-t\xi}. \tag{12}$$

It is straightforward to check that

$$\delta_\alpha A_t = Y_{\alpha_t} A_t = d\alpha_t + [A_t, \alpha_t], \tag{13}$$

where

$$\alpha_t = e^{t\xi}(\alpha + Z_\alpha)e^{-t\xi}, \tag{14}$$

$$\frac{\partial}{\partial t} A_t = -Y_\xi A_t = -(d\xi + [A_t, \xi]) \tag{15}$$

and

$$\frac{\partial}{\partial t} \alpha_t = -[\alpha_t, \xi] - Z_\alpha \xi. \tag{16}$$

We have assumed that the action of Z_α on ξ implied by (8) is well defined, which is the case if ξ is small.

It follows from (9) and (13,14,15,16) that

$$\begin{aligned}
\delta_\alpha W &= \int_0^1 dt[(\delta_\alpha \xi) \cdot G[A_t] + \xi \cdot (\delta_\alpha G[A_t])] \\
&= \int_0^1 dt[(\delta_\alpha \xi) \cdot G[A_t] + \xi \cdot (Y_{\alpha_t} G[A_t])] \\
&= \int_0^1 dt[(Z_\alpha \xi) \cdot G[A_t] + [\alpha_t, \xi] \cdot G[A_t] + \alpha_t \cdot Y_\xi G[A_t]] \\
&= \int_0^1 dt[-(\frac{\partial}{\partial t} \alpha_t) \cdot G[A_t] + \alpha_t \cdot (-\frac{\partial}{\partial t} G[A_t])] \\
&= -\int_0^1 dt \frac{\partial}{\partial t}(\alpha_t \cdot G[A_t]) \\
&= \alpha_0 \cdot G[A_0] - \alpha_1 \cdot G[A_1], \tag{17}
\end{aligned}$$

where $\alpha_0 = \alpha$, $A_0 = A$ and $\alpha_1 = e^\xi(\alpha + Z_\alpha)e^{-\xi}$. Note that α_1 belongs to $\mathbf{h}$ since (8) implies that

$$Z_\alpha e^{-\xi} = -\alpha e^{-\xi} + e^{-\xi}\rho \tag{18}$$

for some $\rho \in \mathbf{h}$. Therefore if $G_\alpha[A]$ also satisfies

$$G_\alpha[A] = 0, \quad \alpha \in \mathbf{h}, \tag{19}$$

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then the term $\alpha_1 \cdot G[A_1]$ vanishes and $W[\xi, A]$ generates the anomaly as in (4).

This $W[\xi, A]$ is invariant under the gauge transformations of H. Its expression (11) is very convenient for computing the vertices involving different numbers of factors ξ. For example, the Bardeen anomaly [7] is considered in [5] and the corresponding effective action is obtained by using (11). It contains the five-pseudoscalar interaction in four space-time dimensions,

$$\frac{2}{15\pi^2 F_\pi^5} \int Tr(\Pi(d\Pi)^4), \tag{20}$$

where $\xi = \Pi/F_\pi$. Note that there is a misprint on p.97 of [5], the numerical factor of $1/6$ there should be replaced by $2/15$.

2.2   Chern-Simon Form and Non-Abelian Anomaly

We consider a $2n$-dimensional space-time. In order to make sense of many formulas given below, we must embed space-time in a manifold with dimension larger than $2n + 1$. All quantities on space-time such as the gauge field $A(x)$ and the group element g are accordingly extended to the higher dimensional manifold. Similarly the exterior differentiation d operates in the higher dimensional manifold.

The Chern-Simon form $\omega_{2n+1}(A), n = 1, 2, ...,$ satisfies

$$Tr F^{n+1} = d\omega_{2n+1}(A). \tag{21}$$

(The existence of $\omega_{2n+1}(A)$ is guaranteed because $d(Tr F^{n+1}) = 0$ due to the Bianchi identity $dF = [F, A]$.) Its form can be obtained, for instance, by using the Cartan homotopy operator k [4, 8] (see Appendix 1) and the one parameter family of gauge fields $A_t = tA, F_t = dA_t + A_t^2 = tF + (t^2 - t)A^2$. Explicitly, it is[4, 8, 9]

$$\omega_{2n+1}(A) = (n + 1) \int_0^1 dt Tr(A F_t^n). \tag{22}$$

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It is easy to see from (7) that the Chern character TrF^{n+1} is gauge invariant,

$$\delta_\alpha TrF^{n+1} = 0, \tag{23}$$

so there exists some $\omega_{2n}^1(\alpha; A)$ such that §

$$\delta_\alpha \omega_{2n+1}(A) = d\omega_{2n}^1(\alpha; A). \tag{24}$$

It is well known[8] that one can take

$$\omega_{2n}^1(\alpha; A) = n(n+1) \int_0^1 dt(1-t)Tr(\alpha d(AF_t^{n-1})). \tag{25}$$

For example, the explicit expressions of the ω's of interest for 4-dimensional physics are

$$\omega_5(A) = Tr(A(dA)^2 + \frac{3}{2}A^3 dA + \frac{3}{5}A^5), \tag{26}$$

$$\omega_4^1(\alpha; A) = Tr(\alpha d(AdA + \frac{1}{2}A^3)). \tag{27}$$

Both are proportional to

$$\frac{1}{2}Tr(\lambda_a\{\lambda_b, \lambda_c\}), \tag{28}$$

the symmetric invariant d-symbol of $\mathbf{g}$.

It follows from (10) and (24) that there exists some $\omega_{2n-1}^2(\alpha_1, \alpha_2; A)$ such that

$$\delta_{\alpha_1}\omega_{2n}^1(\alpha_2; A) - \delta_{\alpha_2}\omega_{2n}^1(\alpha_1; A) - \omega_{2n}^1([\alpha_1, \alpha_2]; A) = d\omega_{2n-1}^2(\alpha_1, \alpha_2; A). \tag{29}$$

Hence, the space-time integral $\int \omega_{2n}^1(\alpha; A)$ satisfies the consistency condition (9). We shall refer to (25) and (27) or their space-time integrals as the canonical anomalies. They have a meaning in $2n$-dimensional space-time which is independent of the extension to higher dimensions.

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§ Following standard notation the subscript denotes the degree of the differential form in dx and the superscript its degree in α. Note that the definitions (21) and (24) do not fix the ω_{2n+1} and ω_{2n}^1 completely: (21) determines the form ω_{2n+1} up to d of something while (24) determines ω_{2n}^1 up to d of something plus δ_α of something.

The anomaly (25) possesses the global symmetry G, but in general it does not vanish on any nontrivial subgroup $H \subset G$. If it did, its global symmetry would imply that it also vanishes on the ideal generated by $\mathbf{h}$. ¶ Therefore a nontrivial anomaly is possible only if $\mathbf{h}$ belongs to a proper ideal of $\mathbf{g}$. Since the global symmetry G is not required in most physical applications, in order to save the local H symmetry one must modify this anomaly by adding global G violating counterterms to the effective action. This is what we will do in the next section.

3   Counterterm Shifting the Anomaly and Effective Action

We want to construct an effective action $W[\xi, A]$ for $\xi \in \mathbf{k}$ with the property that

$$\delta_\alpha W = 0, \quad \alpha \in \mathbf{h}. \tag{30}$$

If we define W by (11), then (30) follows if G_α satisfies (19). Therefore what we need is to find a consistent anomaly $G_\alpha[A]$ which satisfies (19). This will be done in this section.

In the following we will assume that G/H is reductive and that $H \subset G$ is an *anomaly free embedding*. The latter means

$$\omega_{2n+1}(A) = 0, \quad \text{for } A \text{ restricted to } \mathbf{h}. \tag{31}$$

This is equivalent to the condition that the d-symbol for $\mathbf{g}$ has

$$d_{a_1 a_2 \cdots a_{n+1}} \equiv Str(\lambda_{a_1}, \lambda_{a_2}, \cdots, \lambda_{a_{n+1}}) = 0 \tag{32}$$

if all $\lambda_{a_1}, \lambda_{a_2}, \cdots, \lambda_{a_{n+1}} \in \mathbf{h}$. The symmetric trace Str for matrix valued

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¶A Lie subalgebra $\mathbf{I} \subset \mathbf{g}$ is an ideal if $[\mathbf{g}, \mathbf{I}] \subset \mathbf{I}$; the ideal generated by $\mathbf{h}$ is the smallest ideal containing $\mathbf{h}$.

forms C_i is defined by

$$Str(C_1, C_2, \cdots, C_N) = \frac{1}{N!} \sum_P (-1)^{f(P)} Tr(C_{P_1} C_{P_2} \cdots C_{P_N}), \qquad (33)$$

with the sum over all the permutations $P = (P_1, P_2, \cdots, P_N)$ of $(1, 2, \cdots, N)$ and $f(P)$ is the number of times the permutation P permutes two odd objects. In the following $Str(C_1, \cdots, C_1, C_2, \cdots, C_2, C_3, \cdots)$, where C_1 and C_2 appear n_1 and n_2 times respectively, will be abbreviated as $Str(C_1^{n_1}, C_2^{n_2}, C_3, \cdots)$.

Define A_h and A_k by the splitting

$$A = A_h + A_k, \quad A_h \in \mathbf{h}, \quad A_k \in \mathbf{k}, \qquad (34)$$

and write

$$\alpha = \beta + \gamma, \qquad (35)$$

where β and γ are the $\mathbf{h}$ and $\mathbf{k}$ parts of α, respectively.

Since $\mathbf{g} = \mathbf{h} + \mathbf{k}$ is reductive, A_h and A_k transform respectively as

$$\delta_\beta A_h = d\beta + [A_h, \beta] \qquad (36)$$

and

$$\delta_\beta A_k = [A_k, \beta] \qquad (37)$$

for $\beta \in \mathbf{h}$.

Given any one-parameter family A_t and $F_t = dA_t + A_t^2$, one can introduce the Cartan homotopy operator k acting on polynomials in A and F such that

$$kC(A, F) = \int_0^1 l_t C(A_t, F_t) \qquad (38)$$

and

$$(kd + dk)C(A, F) = C(A_1, F_1) - C(A_0, F_0) \qquad (39)$$

for any polynomial C in A, F. (See Appendix 1 for the definition of l_t and other details.)

Now, let A_0, A_1 be two fixed gauge fields and consider the Cartan homotopy operator k defined on the family of gauge fields

$$A_t = tA_1 + (1 - t)A_0. \tag{40}$$

Define

$$B_{2n}(A_0, A_1) = -k\omega_{2n+1}(A). \tag{41}$$

Using (22), it is [||]

$$B_{2n}(A_0, A_1) = n(n + 1) \int \int_\Delta d\mu d\lambda Str(A_0, A_1, F_{\mu,\lambda}^{n-1}), \tag{42}$$

where $F_{\mu,\lambda} = dA_{\mu,\lambda} + A_{\mu,\lambda}^2$, $A_{\mu,\lambda} = \mu A_0 + \lambda A_1$ and the integration is over a triangle Δ in the (μ, λ) plane with vertices $(0, 1), (1, 0)$ and the origin. It is clear from this expression that B_{2n} is the generalization of α_{2n} introduced in [8]

$$\alpha_{2n}(dgg^{-1}, A) = B_{2n}(-dgg^{-1}, A)$$

for any $g \in G$. Now we give the derivation of (42). Since, from (22),

$$\omega_{2n+1}(A) = (n + 1) \int_0^1 dt Tr[A(tF + (t^2 - t)A^2)^n], \tag{43}$$

we shall call the parameter in (40) s instead of t. Then

$$
\begin{aligned}
B_{2n}(A_0, A_1) &= -k\omega_{2n+1}(A) \\
&= -\int_0^1 l_s \omega_{2n+1}(A_s) \\
&= n(n + 1) \int_0^1 dt \int_0^1 ds Str(A_s, t\frac{\partial A_s}{\partial s}, F_{t,s}^{n-1}) \\
&= n(n + 1) \int_0^1 dt \int_0^1 ds Str(\frac{\partial A_{t,s}}{\partial t}, \frac{\partial A_{t,s}}{\partial s}, F_{t,s}^{n-1}), \tag{44}
\end{aligned}
$$

[||] Notice that for 2-dimensions, $B_2(A_0, A_1) = Tr(A_0 A_1)$ and the counterterm $B_2(A_h, A)$ is zero because $Tr(A_h^2) = 0$ (A_h is odd) and $Tr(A_h A_k) = 0$ by orthogonality of **h** and **k**. The canonical anomaly $\omega_2^1(\alpha; A)$ already satisfies (19).

where $A_{t,s} = tA_s$ and

$$
\begin{aligned}
F_{t,s} &= tF_s + (t^2 - t)A_s^2 = tdA_s + tA_s^2 + (t^2 - t)A_s^2 \\
&= d(tA_s) + (tA_s)^2 = dA_{t,s} + A_{t,s}^2.
\end{aligned}
\tag{45}
$$

Now changing variables from (t, s) to (μ, λ) by

$$
\lambda = ts, \quad \mu = t(1 - s),
\tag{46}
$$

the integration goes over the triangle Δ. Notice that

$$
\begin{aligned}
A_{t,s} = A_{\mu,\lambda} &= \mu A_0 + \lambda A_1, \\
dtds\frac{\partial A_{t,s}}{\partial t}\frac{\partial A_{t,s}}{\partial s} &= d\mu d\lambda\frac{\partial A_{\mu,\lambda}}{\partial \mu}\frac{\partial A_{\mu,\lambda}}{\partial \lambda}
\end{aligned}
\tag{47}
$$

and

$$
\frac{\partial A_{\mu,\lambda}}{\partial \mu} = A_0, \quad \frac{\partial A_{\mu,\lambda}}{\partial \lambda} = A_1.
\tag{48}
$$

This completes the proof of (42).

For simplicity, the space-time integration considered below will be over the compactified $2n$-dimensional Euclidean space-time S^{2n}. Define

$$
G_\alpha = \int_{S^{2n}} \tilde{\omega}_{2n}^1(\alpha; A_h, A),
\tag{49}
$$

where

$$
\tilde{\omega}_{2n}^1(\alpha; A_h, A) = \omega_{2n}^1(\alpha; A) + \delta_\alpha B_{2n}(A_h, A),
\tag{50}
$$

i.e.

$$
G_\alpha = G_{0\alpha} + \delta_\alpha \int_{S^{2n}} B_{2n}(A_h, A),
\tag{51}
$$

where

$$
G_{0\alpha}[A] = \int_{S^{2n}} \omega_{2n}^1(\alpha; A).
\tag{52}
$$

We claim that

$$
G_\beta = 0, \quad \beta \in \mathbf{h}.
\tag{53}
$$

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First, in view of (6) and (36), we have for $A_0 = A_h$ and $A_1 = A$,

$$\delta_\beta A_i = d\beta + [A_i, \beta], \quad i = 0, 1, \tag{54}$$

so

$$\delta_\beta A_t = d\beta + [A_t, \beta] \tag{55}$$

for $A_t = tA_1 + (1 - t)A_0$. It is not hard to see that

$$l_t \delta_\beta = \delta_\beta l_t. \tag{56}$$

Indeed

$$
\begin{aligned}
l_t \delta_\beta A_t &= \delta_\beta l_t A_t = 0, \\
l_t \delta_\beta F_t &= l_t(F_t \beta - \beta F_t) \\
&= dt[A_1 - A_0, \beta]
\end{aligned}
$$

and

$$
\begin{aligned}
\delta_\beta l_t F_t &= dt \delta_\beta (A_1 - A_0) \\
&= dt[A_1 - A_0, \beta].
\end{aligned}
$$

Therefore

$$k\delta_\beta = \delta_\beta k \tag{57}$$

and so

$$
\begin{aligned}
\delta_\beta B_{2n}(A_h, A) &= -k\delta_\beta \omega_{2n+1}(A) \\
&= -kd\omega_{2n}^1(\beta; A) \\
&= -(kd + dk)\omega_{2n}^1(\beta; A) + dk\omega_{2n}^1(\beta; A) \\
&= -\omega_{2n}^1(\beta; A) + \omega_{2n}^1(\beta; A_h) + dk\omega_{2n}^1(\beta; A). \tag{58}
\end{aligned}
$$

The first term cancels the canonical anomaly, the second term is zero due to (32) and the third integrates to zero. Hence we have proved (53).

For four dimensions, $n = 2$, (42) gives

$$B_4(A_h, A) = \frac{1}{2}Tr[(A_hA - AA_h)(F + F'_h) + AA_h^3 - A_hA^3 + \frac{1}{2}A_hAA_hA], \quad (59)$$

where $F'_h = dA_h + A_h^2$ and $F = dA + A^2$. It is very easy, using (54), to check directly that (59) satisfies (58). Notice that all commutator terms from (54) will cancel under the trace, so it is sufficient to keep the terms with $d\beta$, which makes the computation very easy.

To get an explicit formula for $\tilde{\omega}_{2n}^1(\gamma; A_h, A), \gamma \in \mathbf{k}$, introduce

$$\tilde{\omega}_{2n+1}(A_0, A_1) = (n + 1) \int_0^1 dt Tr((A_1 - A_0)F_t^n) \quad (60)$$

for $A_t = tA_1 + (1 - t)A_0$. Using (39) and (41), it is not hard to check that

$$\tilde{\omega}_{2n+1}(A_0, A_1) = \omega_{2n+1}(A_1) - \omega_{2n+1}(A_0) + dB_{2n}(A_0, A_1). \quad (61)$$

In fact

$$
\begin{aligned}
&\omega_{2n+1}(A_1) - \omega_{2n+1}(A_0) + dB_{2n}(A_0, A_1) \\
&= \omega_{2n+1}(A_1) - \omega_{2n+1}(A_0) - dk\omega_{2n+1}(A) \\
&= \omega_{2n+1}(A_1) - \omega_{2n+1}(A_0) - (kd + dk)\omega_{2n+1}(A) + kd\omega_{2n+1}(A) \\
&= kTrF^{n+1} \\
&= (n + 1) \int_0^1 dt Tr((A_1 - A_0)F_t^n).
\end{aligned} \quad (62)
$$

In particular, we have

$$\tilde{\omega}_{2n+1}(A_h, A) = \omega_{2n+1}(A) + dB_{2n}(A_h, A), \quad (63)$$

where we have used again (31). It is clear from (63) that

$$TrF^{n+1} = d\tilde{\omega}_{2n+1}(A_h, A), \quad F = dA + A^2 \quad (64)$$

and

$$\delta_\alpha\tilde{\omega}_{2n+1}(A_h, A) = d\tilde{\omega}_{2n}^1(\alpha; A_h, A). \quad (65)$$

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Now we can extract $\tilde{\omega}_{2n}^1$ from (65) using (60). Although (50) is not of this form, we can choose $\tilde{\omega}_{2n}^1(\gamma; A_h, A)$ to be of the form $Tr(\gamma Q)$ up to exact terms. With this choice it is enough, in order to find Q, to collect the $d\gamma$ terms in $\delta_\gamma \tilde{\omega}_{2n+1}(A_h, A)$. For $\gamma \in \mathbf{k}$, it is

$$\delta_\gamma A = d\gamma + [A, \gamma] \tag{66}$$

and

$$\begin{align}
\delta_\gamma A_h &= [A_k, \gamma]_h = [A_k, \gamma] - u, \tag{67} \\
\delta_\gamma A_k &= d\gamma + [A_h, \gamma] + u, \tag{68} \\
\delta_\gamma F_h &= [F_k, \gamma] - du - [A_h, u]_+ + [A_k, u]_+, \tag{69} \\
\delta_\gamma F_k &= [F_h, \gamma] + du + [A_h, u]_+ - [A_k, u]_+, \tag{70}
\end{align}$$

where

$$u = [A_k, \gamma]_k \tag{71}$$

is the $\mathbf{k}$ component of $[A_k, \gamma]$ and

$$F_k = dA_k + A_h A_k + A_k A_h, \quad F_h = dA_h + A_h^2 + A_k^2. \tag{72}$$

Noticing that $F_t = F_h + tF_k + (t^2 - 1)A_k^2$, we obtain in agreement with [4], where the result was obtained by a different argument,

$$\tilde{\omega}_{2n}^1(\gamma; A_h, A) = (n+1) \int_0^1 dt \; [\; Tr(\gamma F_t^n) + n(t^2 - 1)Str(A_k, [A_k, \gamma], F_t^{n-1}) +$$
$$+ n(1 - t)Str(A_k, u, F_t^{n-1})]. \tag{73}$$

This is a general formula for the shifted anomaly in any number of dimensions. It can be used in (11) to obtain the effective action. In section 4, instead of using (73), we shall compute explicitly the shifted anomaly for four dimensions by using (50) and (59) directly. This will exhibit an interesting feature of the anomaly in the case of a symmetric space, see (91) below and the remark immediately after it.

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Finally we point out that $\tilde{\omega}_{2n+1}(A_0, A_1)$ is invariant under simultaneous gauge transformations of A_0 and A_1

$$\tilde{\omega}_{2n+1}((A_0)^g, (A_1)^g) = \tilde{\omega}_{2n+1}(A_0, A_1) \tag{74}$$

for any $g \in G$ where

$$A^g = g^{-1}(A + d)g, \quad g \in G. \tag{75}$$

This is clear from its definition (60).

To get the effective action, we use (11) to integrate the anomaly (51). Using (14), we have

$$\int_0^1 dt \int_{S^{2n}} Y_\xi B_{2n}((A_t)_h, A_t) = -\int_{S^{2n}} \int_0^1 dt \frac{\partial}{\partial t} B_{2n}((A_t)_h, A_t)$$

$$= -\int_{S^{2n}} B_{2n}((A^g)_h, A^g) + \int_{S^{2n}} B_{2n}(A_h, A),$$

where $A_t = e^{t\xi}Ae^{-t\xi} + e^{t\xi}de^{-t\xi}$, $g = e^{-\xi}$ and $A^g = (A^g)_h + (A^g)_k$ is the splitting $(A^g)_h \in \mathbf{h}, (A^g)_k \in \mathbf{k}$. Therefore,

$$W[\xi, A] = \int_0^1 ds\xi \cdot G_0[A_s] - \int_{S^{2n}} B_{2n}((A^g)_h, A^g) + \int_{S^{2n}} B_{2n}(A_h, A). \tag{76}$$

Using (25), the first term is

$$n(n+1) \int_{S^{2n}} \int_0^1 ds \int_0^1 dt(1-t)Tr(\xi d(A_s F_{t,s}^{n-1})), \tag{77}$$

where $F_{t,s} = dA_{t,s} + A_{t,s}^2$ with $A_{t,s} = tA_s$. As explained at the end of Appendix 2, (77) is equal to the simpler expression, in agreement with [8],

$$\int_{S^{2n}} L_0 - \int_{S^{2n}} B_{2n}(-dgg^{-1}, A), \tag{78}$$

where

$$L_0 = (-1)^n \frac{n!(n+1)!}{(2n)!} \int_0^1 dt Tr(\xi U_t^{2n}), \tag{79}$$

$g = e^{-\xi}$ and

$$U_t = e^{t\xi}de^{-t\xi}. \tag{80}$$

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It is obvious that

$$U_t^{2n} = -dU_t^{2n-1}. \tag{81}$$

When all gauge fields vanish, the surviving effective action is given by the Lagrangian density $L = L_0 - B_{2n}(U_h, U)$, where $U = g^{-1}dg = U_h + U_k$ for $U_h \in \mathbf{h}, U_k \in \mathbf{k}$. It can be checked that L changes by an exact term under a global G transformation. Therefore the action $\int_{S^{2n}} L$ is invariant.

Together with the standard (nonlinear) kinetic energy term [13] for ξ, $W[\xi, A]$ gives the interaction of the Nambu-Goldstone bosons in the presence of external gauge fields A. It is perhaps worthwhile to emphasize again that $W[\xi, A]$ is not globally G-invariant: the Bardeen counterterm B_{2n} breaks the invariance explicitly. In the case of $G = SU(3) \times SU(3), H = SU(3)_V$ [5], $W[\xi, A]$ can be used, in the approximation of vector meson dominance, to estimate the vertices which describe the strong interactions between vector , axial vector and pseudoscalar mesons. It is the shifted form of the anomaly which agrees reasonably well with experiment [14].

4   Parity and Effective Actions

The result for $W[\xi, A]$ in the last section is not the only possible one with the property that the anomaly $G_\alpha[A]$ vanishes for $\alpha \in \mathbf{h}$. If we have a Lie algebra isomorphism we can decompose the canonical anomaly ω_{2n}^1 and B_{2n} further according to their eigenvalues under the isomorphism. For physical applications, very often one has to choose $W[\xi, A]$ to respect all symmetry properties of the physical system. For example, the effective action of the mesons should be even under parity transformation and even under charge conjugation. Here we consider the generalization of the parity transformation for symmetric spaces, and find effective actions with definite parity in addition to the local H symmetry.

For symmetric spaces (2), one can define the "parity" transformation P

that corresponds to an isomorphism of the Lie algebra

$$\mathbf{h} \rightarrow \mathbf{h}, \quad \mathbf{k} \rightarrow -\mathbf{k}, \tag{82}$$

so that

$$PA_h = A_h,$$
$$PA_k = -A_k,$$
$$P\xi = -\xi,$$
$$P\alpha = P(\beta + \gamma) = \beta - \gamma. \tag{83}$$

It is $P\delta_\alpha = \delta_\alpha P$ for all $\alpha \in \mathbf{g}$. One can split the counterterm B ** and the canonical anomaly ω,

$$B = B_+ + B_-, \quad B_\pm = (1 \pm P)B/2, \tag{84}$$
$$\omega = \omega_+ + \omega_-, \quad \omega_\pm = (1 \pm P)\omega/2. \tag{85}$$

For four dimensions, they are ††

$$\omega_+(\alpha) = -Tr(d\beta f_+ + d\gamma f_-), \quad \omega_-(\alpha) = -Tr(d\beta f_- + d\gamma f_+), \tag{86}$$

where

$$f_+ = A_h dA_h + A_k dA_k + \tfrac{1}{2}(A_h^3 + A_h A_k^2 + A_k A_h A_k + A_k^2 A_h),$$
$$f_- = A_h dA_k + A_k dA_h + \tfrac{1}{2}(A_k^3 + A_k A_h^2 + A_h A_k A_h + A_h^2 A_k) \tag{87}$$

and

$$B_+ = Tr[\tfrac{1}{2}(A_h A_k - A_k A_h)F_k - \tfrac{1}{4}A_h A_k A_h A_k], \tag{88}$$
$$B_- = Tr[(A_h A_k - A_k A_h)F_h - \tfrac{3}{2}A_h A_k^3 - \tfrac{1}{2}A_h^3 A_k]. \tag{89}$$

** Here we denote B_{2n} and ω_{2n}^1 as B and ω for simplicity.

†† In this section we will not write down the unimportant exact terms which integrate to zero and will not contribute to the anomaly. Notice that, since B and ω are 4-forms, the space-time integrals of B_- and ω_- are even and those of B_+ and ω_+ are odd under the physical parity operation which changes the sign of some fields and the space part of x in all fields.

For $\alpha = \beta$, it is

$$\omega_+(\beta) + \delta_\beta B_+ = -Tr[d\beta(A_h dA_h + \frac{1}{2}A_h^3)] \tag{90}$$

and

$$\omega_-(\beta) + \delta_\beta B_- = 0. \tag{91}$$

Notice that one can always shift the anomaly ω_- to vanish on **h** by using the counterterm B_-, we don't have to assume that $H \subset G$ is an anomaly free embedding.

For $\alpha = \gamma$, using (66-72) for $u = 0$, one finds

$$\omega_-(\gamma) + \delta_\gamma B_- = Tr[\gamma(3F_h^2 + F_k^2 - 4(A_k^2 F_h + A_k F_h A_k + F_h A_k^2) + 8A_k^4)], \tag{92}$$

where $F_h = dA_h + A_h^2 + A_k^2$ and $F_k = dA_k + A_h A_k + A_k A_h$. This generalizes the result of Bardeen [7]. In his paper he considered essentially $G = U(N) \times U(N)$ for which the trace of parity-even terms vanish (because $Tr = tr_L - tr_R$) and therefore $\omega_+(\alpha) + \delta_\alpha B_+ = 0$ automatically. For a general symmetric space G/H, one finds,

$$\omega_+(\gamma) + \delta_\gamma B_+ = \frac{3}{2}Tr[\gamma(F_h F_k + F_k F_h - F_k A_k^2 - A_k F_k A_k - A_k^2 F_k)]$$
$$+ Tr[[A_k, \gamma][F_h', A_h]_+] - \frac{1}{4}Tr[[A_k, \gamma][A_h, A_h^2]_+]. \tag{93}$$

The last line vanishes because the d-symbol is zero when restricted to **h** and one obtains the following non-trivial anomaly,

$$\omega_+(\gamma) + \delta_\gamma B_+ = \frac{3}{2}Tr[\gamma(F_h F_k + F_k F_h - F_k A_k^2 - A_k F_k A_k - A_k^2 F_k)]. \tag{94}$$

Using $G_{\alpha\pm}(A) = \int_{S^{2n}}(\omega_\pm(\alpha; A) + \delta_\alpha B_\pm)$, one can get the corresponding parity-definite effective action by (11). They agree with those obtained by doing the parity splitting directly on $W[\xi, A]$.

Notice that both $\omega_\pm$ satisfy the consistency condition and both are non-trivial, in the sense that they cannot be written as the δ_α of something local.

It is well known that the canonical anomaly (27) arises when Weyl fermions are coupled to the external gauge fields, it is natural to ask what are the corresponding fermion actions that give rise to $\omega_{\pm}$. Consider the Lagrangian

$$\mathcal{L}(\psi_1, \psi_2, A_h, A_k) = i\overline{\psi_1}(\partial\!\!\!/ + \mathcal{A}_+)\psi_1 + i\overline{\psi_2}(\partial\!\!\!/ + \mathcal{A}_-)\psi_2, \qquad (95)$$

where

$$\mathcal{A}_- = A_h - A_k, \quad \mathcal{A}_+ = A_h + A_k. \qquad (96)$$

It is obvious that $\mathcal{L}$ is invariant under the infinitesimal transformation

$$\delta_\alpha \psi_1 = (\beta + \gamma)\psi_1,$$
$$\delta_\alpha \mathcal{A}_+ = d\beta + d\gamma + [\mathcal{A}_+, \beta + \gamma],$$
$$\delta_\alpha \psi_2 = (\beta - \gamma)\psi_2,$$
$$\delta_\alpha \mathcal{A}_- = d\beta - d\gamma + [\mathcal{A}_-, \beta - \gamma] \qquad (97)$$

for arbitrary $\beta \in \mathbf{h}, \gamma \in \mathbf{k}$. One then obtains ω_- from $\mathcal{L}(\psi_R, \psi_L, A_h, A_k)$ for a set of left-handed spinors ψ_L and a set of right-handed spinors ψ_R and ω_+ from $\mathcal{L}(\psi_R, \psi'_R, A_h, A_k)$ for 2 sets of right-handed spinors ψ_R, ψ'_R.

Finally, we give the explicit form of the anomaly which vanishes on $\mathbf{h}$ for the reductive case (1) in 4 dimensions. Define $\delta' = \delta_\gamma - \delta_\gamma^{symm}$, where δ_γ acts on the gauge fields according to (66)-(72) and δ_γ^{symm} is given by the corresponding formula with u set equal to zero. Then

$$\delta' A_h = -u, \quad \delta' A = 0,$$
$$\delta' F_h = -du - [A_h, u]_+ + [A_k, u]_+, \quad \delta' F = 0. \qquad (98)$$

Since $\delta_\gamma^{symm} B$ has already been given in (92) and (93), it is enough to calculate $\delta' B$. It follows from (59) straightforwardly that

$$\delta' B = -\frac{1}{2} Tr[u([F, A_k]_+ + 2[F'_h, A]_+ - A_h^3 - A_k^3)]. \qquad (99)$$

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Adding (92), (93) and (99), one obtains the full answer for the shifted anomaly for the reductive case in four space-time dimensions

$$\tilde{\omega} = \omega(\gamma) + \delta_\gamma B =$$
$$Tr[\gamma(3F_h^2 + F_k^2 - 4(A_k^2 F_h + A_k F_h A_k + F_h A_k^2) + 8A_k^4)] +$$
$$\frac{3}{2} Tr[\gamma(F_h F_k + F_k F_h - F_k A_k^2 - A_k F_k A_k - A_k^2 F_k)] +$$
$$-\frac{1}{2} Tr[u([F + 2F_h', A_k]_+ - A_k^3)]. \tag{100}$$

Here we have kept that part in the last line of (93) which does not vanish when G/H is not symmetric. This expression (100) is the one to be used in (49) and (11) for the general reductive case.

5   Acknowledgement

6   Appendix 1

For the sake of completeness, we shall review the definition and basic properties of the Cartan homotopy operator k in this Appendix. Consider the graded algebra $\mathcal{G}$ spanned by the one-form A and the two-form dA. Equivalently, one can use the generators A and $F = dA + A^2$. The exterior derivative d increases the degree of a form by 1. We want to introduce a second operator k on $\mathcal{G}$ so that it decreases the degree of a form by 1 and behaves in some sense like "d^{-1}".

For any one-parameter family of gauge fields A_t and $F_t = dA_t + A_t^2$, define an operator l_t as

$$l_t A_t = 0, \quad l_t F_t = d_t A_t = dt \frac{\partial A_t}{\partial t} \tag{101}$$

and extend it to an arbitrary polynomial $C(A_t, F_t)$ as a derivation or anti-derivation depending on whether we take dt to be odd or even. Here we shall take dt to be odd and so l_t acts as a derivation. The Cartan homotopy operator k [4, 8] is defined on any element of $\mathcal{G}$ which is a polynomial $C(A, F)$ as

$$kC(A, F) = \int_0^1 l_t C(A_t, F_t). \tag{102}$$

We will treat $\int_0^1$ as odd since the combination $\int_0^1 dt$ is even. We take the convention that we move dt next to $\int_0^1$ before integrating.

It is easy to check that

$$l_t d - d l_t = d_t \tag{103}$$

on $\mathcal{G}$. Indeed,

$$
\begin{aligned}
(l_t d - d l_t) A_t &= l_t (F_t - A_t^2) = d_t A_t, \\
(l_t d - d l_t) F_t &= l_t (F_t A_t - A_t F_t) - d(d_t A_t) \\
&= (d_t A_t) A_t - A_t d_t A_t + d_t (dA_t) \\
&= d_t F_t
\end{aligned}
$$

and so

$$
\begin{aligned}
(kd + dk) C(A, F) &= \int_0^1 (l_t d - d l_t) C(A_t, F_t) \\
&= \int_0^1 d_t C(A_t, F_t) \\
&= C(A_1, F_1) - C(A_0, F_0). \tag{104}
\end{aligned}
$$

In the above we have used the Bianchi identity

$$dF_t = F_t A_t - A_t F_t.$$

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As an application of (104), (22) is obtained from (21) for $A_t = tA$. It is

$$d\omega_{2n+1}(A) = Tr F^{n+1} = (kd + dk)Tr F^{n+1} = dkTr F^{n+1}. \qquad (105)$$

Hence

$$\omega_{2n+1}(A) = kTr F^{n+1} = (n+1)\int_0^1 dt Tr(AF_t^n). \qquad (106)$$

These considerations can be extended to a multi-parameter family of gauge fields [4] but we do not need this generalization here.

7 Appendix 2

In this Appendix, we describe and extend the results of the main text in more geometric terms. The elements of G/H are equivalence classes of elements of G. In each class we choose an element $g \in G$ to represent that class. This determines a subset of G which is in one-to-one correspondence with G/H. If g is sufficiently close to the identity of G, it can be parametrized as

$$g = e^{-\xi}, \qquad (107)$$

where $\xi \in \mathbf{k}$ [13]. However, most of the formulas of this Appendix have more general validity.

Denote by D_{2n+1} a $(2n+1)$-dimensional disk with its boundary fixed by the compactified space-time S^{2n}. We extend the gauge field $A(x)$ and the group element $g(x)$ from S^{2n} to D_{2n+1}. Note that g can always be extended to D_{2n+1} when the homotopy group $\pi_{2n}(G/H)$ is trivial, which we shall assume. The exterior differentiation d operates in D_{2n+1} as well.

Consider the effective action of g in the presence of A

$$W[g, A] = \int_{D_{2n+1}} (\tilde{\omega}_{2n+1}(A_h, A) - \tilde{\omega}_{2n+1}((A^g)_h, A^g)), \qquad (108)$$

where

$$A^g = g^{-1}Ag + g^{-1}dg \qquad (109)$$

and $(A^g)_h \in \mathbf{h}, (A^g)_k \in \mathbf{k}$ is the splitting of $A^g = (A^g)_h + (A^g)_k$. For simplicity, we will denote them as A_h^g and A_k^g in the following. It is, in agreement with (18),

$$\delta_\alpha g = -\alpha g + g\rho \tag{110}$$

for a suitable $\rho \in \mathbf{h}$. It is easy to check that A^g and A_h^g transform as

$$\delta_\alpha A^g = d\rho + [A^g, \rho] \tag{111}$$

and

$$\delta_\alpha A_h^g = d\rho + [A_h^g, \rho]. \tag{112}$$

So the term $\tilde{\omega}_{2n+1}(A_h^g, A^g)$ is invariant due to (74) and, from (65), the variation of W in (108) is the shifted anomaly

$$\delta_\alpha W[g, A] = \delta_\alpha \int_{D_{2n+1}} \tilde{\omega}_{2n+1}(A_h, A) = \int_{S^{2n}} \tilde{\omega}_{2n}^1(A_h, A). \tag{113}$$

Since the integrand of (108) is exact

$$d(\tilde{\omega}_{2n+1}(A_h, A) - \tilde{\omega}_{2n+1}(A_h^g, A^g)) = TrF^{n+1} - Tr(g^{-1}Fg)^{n+1} = 0, \tag{114}$$

where (64) was used, (108) depends only on the values of the integrand on the boundary S^{2n} if $\pi_{2n+1}(G/H)$ is trivial. Otherwise, the coefficient of the action gets quantized [10]. For a more detailed discussion of special situations see [1, 2].

A more explicit expression for $W[g, A]$ (108) can be obtained by using (63)

$$W[g, A] = \int_{D_{2n+1}} (-\tilde{\omega}_{2n+1}(A_h^g, A^g) + \omega_{2n+1}(A)) + \int_{S^{2n}} B_{2n}(A_h, A). \tag{115}$$

Let us consider (115) term by term. The first term is gauge-invariant, but its integrand is in general not closed. The second term is there so that the sum of the integrands of the first two terms is closed. The variation of the second term gives the canonical anomaly G_0 as in (52).

Using (61), we see that

$$\tilde{\omega}_{2n+1}(A_h^g, A^g) = \omega_{2n+1}(A^g) + dB_{2n}(A_h^g, A^g),$$
$$\tilde{\omega}_{2n+1}(-dgg^{-1}, A) = \omega_{2n+1}(A) - \omega_{2n+1}(-dgg^{-1}) + dB_{2n}(-dgg^{-1}, A),$$

where we have used the anomaly free condition for **h**. Noticing that $\omega_{2n+1}(-dgg^{-1}) = -\omega_{2n+1}(g^{-1}dg)$ and $\tilde{\omega}_{2n+1}(-dgg^{-1}, A) = \tilde{\omega}_{2n+1}(0, A^g) = \omega_{2n+1}(A^g)$ due to (74), we can combine the above two equations to rewrite the first two terms in $W[g, A]$ more explicitly as

$$\int_{D_{2n+1}} (-\tilde{\omega}_{2n+1}(A_h^g, A^g) + \omega_{2n+1}(A)) = \int_{D_{2n+1}} [-\omega_{2n+1}(U)]$$
$$- \int_{S^{2n}} [B_{2n}(A_h^g, A^g) + B_{2n}(-dgg^{-1}, A)], \tag{116}$$

where $U = g^{-1}dg$.

The Wess-Zumino-Witten term $\mathcal{L}_{WZW}$ [1, 2, 3, 4, 5, 8, 10] is the only surviving term in the effective action when all gauge fields are set to zero. It is

$$\mathcal{L}_{WZW} = -\omega_{2n+1}(U) - dB_{2n}(U_h, U), \tag{117}$$

where $U = g^{-1}dg = U_h + U_k$ for $U_h \in \mathbf{h}, U_k \in \mathbf{k}$. It is easy to see that

$$\mathcal{L}_{WZW} = -\tilde{\omega}_{2n+1}(U_h, U)$$
$$= -(C_{n+1}^{2n+1})^{-1} \sum_{j=0}^{n} (-1)^j C_{n-j}^{2n+1} Str(f_h^{n-j}, U_k^{2j+1}), \tag{118}$$

where $C_m^n = \frac{n!}{m!(n-m)!}$ is the binomial coefficient and $f_h = dU_h + U_h^2$. For $\mathcal{L}_{WZW}$ to be nonzero, we need $\dim(G/H) \geq 2n + 1$.

In particular for 4-dimensional space-time, one finds, in agreement with [1],

$$\mathcal{L}_{WZW} = -\frac{1}{10} Tr(U_k^5 - 5U_k^3 f_h + 10U_k f_h^2). \tag{119}$$

The integral $\int_{D_{2n+1}} [-\omega_{2n+1}(U)]$ in (116) can be reduced to a $2n$-dimensional space-time integral. It is

$$- \omega_{2n+1}(U) = (-1)^{n-1} \frac{n!(n+1)!}{(2n+1)!} Tr(U^{2n+1}). \tag{120}$$

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As explained above, when g is sufficiently close to the identity of G, the Nambu-Goldstone bosons ξ are given by (107). A convenient choice of coordinates for D_{2n+1} is (t,x) with $t \in [0,1]$ and $x \in S^{2n}$. The group element becomes a function $g(t,x)$ with $g(1,x) = e^{-\xi(x)}$ and $g(0,x) = 1$. The differentiation operator d in $2n+1$ dimensions becomes now

$$d = d_t + d_x, \tag{121}$$

where $d_t = dt\frac{\partial}{\partial t}, d_x = dx^i\frac{\partial}{\partial x^i}$. We have

$$
\begin{aligned}
U &= g^{-1}dg \\
&= g^{-1}d_xg + g^{-1}d_tg \\
&= U_t + dtg^{-1}\partial_tg,
\end{aligned} \tag{122}
$$

where $U_t = g^{-1}(t,x)d_xg(t,x)$. Since $(dt)^2 = 0$,

$$TrU^{2n+1} = (2n+1)dtTr(g^{-1}\partial_tg\, U_t^{2n}). \tag{123}$$

A particular convenient extension for the group element is

$$g(t,x) = e^{-t\xi(x)}. \tag{124}$$

This corresponds to (11). With this choice,

$$-\omega_{2n+1}(U) = (-1)^n\frac{n!(n+1)!}{(2n)!}dtTr(\xi U_t^{2n}), \tag{125}$$

where $U_t = e^{t\xi}d_xe^{-t\xi}$. In (80) and (81) we have simply written d for d_x. The advantage of integrating over D_{2n+1} is that $W[\xi, A]$ as given in (76) and (77) can be written compactly as (108) with $g = e^{-\xi}$ and be given a geometrical meaning.

The differential forms $\omega_{2n+1}(U)$, or equivalently $\tilde{\omega}_{2n+1}(U_h, U)$ from (117,118), are generators of the cohomology groups of G/H [1, 2]. Suitably normalized, their integrals over cycles in G/H will give integers which are topological invariants of G/H and can be related to invariants in the combinatorial topology of G/H. When the anomaly arises in perturbation theory from fermion

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loops its normalization should agree with that determined geometrically. As mentioned in the Introduction, the correct normalization has been discussed for most cases of interest in [1, 8, 9, 10, 11, 12].

References

[1] E. D'Hoker and S. Weinberg, *General Effective Actions*, Phys. Rev. D 50 (1995) R6050.

[2] E. D'Hoker, *Invariant Effective Actions, Cohomology of Homogeneous Spaces and Anomalies*, Nucl. Phys. B 451 (1995) 725;
E. D'Hoker, *Invariant Effective Actions and Cohomology*, UCLA/95/TEP/11.

[3] Y.-S. Wu, *Cohomology of Gauge Fields and Explicit Construction of Wess-Zumino Lagrangians in Nonlinear Sigma Models over G/H*, Phys. Lett. 153 B (1985) 70.

[4] J. Mañes, R. Stora and B. Zumino, *Algebraic Study of Chiral Anomalies*, Comm. Math. Phys. 102 (1985) 157.

[5] J. Wess and B. Zumino, *Consequences of Anomalous Ward Identities*, Phys. Lett. 37 B (1971) 95.

[6] H. Georgi and S.L. Glashow, *Gauge Theory without Anomalies*, Phys. Rev. D (1972) 429.

[7] W. A. Bardeen, *Anomalous Ward Identities in Spinorial Field Theories*, Phys. Rev. 184 (1969) 1848.

[8] B. Zumino, *Chiral Anomalies and Differential Geometry*, in "Relativity, Groups and Topology II, Les Houches 1983", eds. B.S. DeWitt and R.

Stora (North-Holland, Amsterdam, 1984). Reprinted in "Current Algebra and Anomalies", eds. S.B. Treiman, R. Jackiw, B. Zumino and E. Witten (Princeton Univ. Press, 1985).

[9] B. Zumino, *Cohomology of Gauge Groups: Cocycles and Schwinger Terms*, Nucl. Phys. B 253 (1985) 477.

[10] E. Witten, *Global Aspects of Current Algebra*, Nucl. Phys. B 223 (1983) 422.

[11] B. Zumino, Y.-S. Wu and A. Zee, *Chiral Anomalies, Higher Dimensions, and Differential Geometry*, Nucl. Phys. B 239 (1984) 477.

[12] R. Bott and R. Seeley, *Some Remarks on the Paper of Callias*, Comm. Math. Phys. 62 (1978) 235.

[13] S. Coleman, J.Wess and B. Zumino, *Structure of Phenomenological Lagrangians. I*, Phys. Rev. 177 (1969) 2239;
C.G. Callan, S. Coleman, J. Wess and B. Zumino, *Structure of Phenomenological Lagrangians. II*, Phys. Rev. 177 (1969) 2247.

[14] Ö. Kaymakcalan, S. Rajeev and J. Schechter, *Non-Abelian Anomaly and Vector Meson Decays*, Phys. Rev. D 30 (1984) 594;
J. L. Petersen, *Non-Abelian Chiral Anomalies and Wess-Zumino Effective Actions*, Acta Physica Polonica B 16 (1985) 271 and references therein.